ON ORDERABLE SET FUNCTIONS AND CONTINUITY I[†]

BY

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ABSTRACT

A set function v (which is not necessarily additive) on a measurable space I is called orderable if for each measurable order $\mathscr{R}$ on I there is a measure $\varphi^{\mathscr{R}} v$ on I such that for all subsets J of I that are initial segments, $\varphi^{\mathscr{R}} v\,(J) = v(J)$. Properties such as nonatomicity, nullness of sets, and weak continuity are shown to be inherited from orderable set functions v to $\varphi^{\mathscr{R}} v$ and vice versa. A characterization of set functions which are absolutely continuous (with respect to some positive measure) in the set of orderable set functions is also given.

Introduction

A set function v (not necessarily additive) on a measurable space I is called *orderable* [1, Sect. 12] if, for each measurable order $\mathscr{R}$ on I there is a measure $\varphi^{\mathscr{R}} v$ on I such that for all subsets J of I that are initial segments in the order $\mathscr{R}$, we have

$$(\varphi^{\mathscr{R}} v)(J) = v(J).$$

To understand orderability intuitively, think of I as consisting of an inhomogeneous liquid, and of $v(S)$ as representing some measure of the worth of a particular part S of I. Think of this liquid as flowing from one place to another,

† Reproduction of this report was partially supported by the Office of Naval Research under contract N-000 14-67-0112-0011; The U.S. Atomic Energy Commission contract AT (04-3)-326-PA # 18; and The National Science Foundation, Grant GP 31393X. Reproduction in whole or in part is permitted for any purposes of the United States Government. This document has been approved for public release and sale; its distribution is unlimited. Reseaıch of this report was carried out at Tel Aviv University.

Received Febıuary 2, 1973

the drops arriving in the order $\mathscr{R}$. As it arrives, each drop of the liquid contributes to (or detracts from) the worth of that portion of the liquid already at the destination. Intuitively, $(\varphi^{\mathscr{R}}v)(S)$ is the total increment contributed in this way by all the drops in a set S. Since v is in general not additive, $\varphi^{\mathscr{R}}v$ will depend strongly on $\mathscr{R}$; in fact, it may not even exist for all $\mathscr{R}$. Orderable v are those for which it does exist.

The reader is referred to [1, Chapt. II] for an explanation of how this notion is motivated by game-theoretic considerations.

It is the purpose of this paper to investigate the properties of the space *ORD* of orderable set functions. We establish in Sections 5 and 6 that for $v \in ORD$, certain properties (such as nonatomicity and weak continuity, defined in Section 3), are inherited by $\varphi^{\mathscr{R}}v$ from v. In Section 7 we show that a set function is absolutely continuous [1, Sect. 6] if and only if $\{\varphi^{\mathscr{R}}v\}$ is weak sequentially compact in the space of all σ-additive measures on I.

1. Notational conventions

Throughout the paper the symbol $\| \quad \|$ for a norm is used in different senses but its meaning is clear each place where it is used.

It is important to distinguish between functions and their values. For example, if μ is a measure, then $\| \mu \|$ is its total variation, whereas $\| \mu(S) \|$ is the absolute value of the number $\mu(s)$.

Composition will usually be denoted by $\circ$; thus if f is defined on the range of μ, then the function whose value on S is $f(\mu(S))$ will be denoted $f \circ \mu$. In the case of composition of linear operators, the symbol $\circ$ will be omitted.

The symbol $\subset$ will be used for inclusion. Set theoretical subtraction will be denoted by $\backslash$, whereas "$-$" will be reserved for algebraic subtraction. ∇ stands for symmetric set subtraction. $f | A$ means f restricted to A. A^c means the complement of A in an appropriate space; if it is unclear which space is meant, we will clarify.

A *measure* is an additive real-valued set function defined on a field which vanishes on $\emptyset$. It will always be specified whether we mean a finitely additive or a σ-additive measure. A probability measure is a nonatomic, σ-additive measure whose value on the entire space is 1. $|\mu|(S)$ means the total variation of μ on S. When μ is additive it is known that $|\mu|$ is additive too. See, for example, [2, III-1-6, p. 98]).

The origin of a linear space will be denoted by 0.

Finally, if $x, y \in E^n$ then $x \leqq y$ iff $x_i \leqq y_i$ for each $1 \leqq i \leqq n$.

2. Basic definitions and conventions

This section summarizes definitions, conventions, and results of [1] which we shall need.

Let $(I, \mathscr{C})$ be the measurable space consisting of the unit interval and the Borel subsets. (This is assumed for simplicity only. All the results remain true if $(I, \mathscr{C})$ is any countably generated and separated Borel space.) A *set function* is a real-valued function v on $\mathscr{C}$ such that $v(\emptyset) = 0$. By a *carrier* of a set function v, we mean a set I' such that $v(S) = v(S \cap I')$ for each $S \in \mathscr{C}$. A set is *null* (or v-null) if it is the complement of a carrier. A set function is *nonatomic* if $\{s\}$ is null for each $s \in I$. (The definition of nonatomicity in this paper differs from the one appearing in [1]; however, it will be proved in [4] that for orderable set functions the two definitions coincide.) A set function is *monotonic* if $S \subset T$ implies $v(S) \leqq v(T)$. The difference between two monotonic set functions is said to be of *bounded variation*. The set of all set functions of bounded variation forms a linear space, which will be called BV. The linear subspace of BV consisting of all bounded, finitely additive, set functions will be denoted FA. Note that $\mu \in FA$ is monotonic iff $\mu(S) \geqq 0$ for all S in $\mathscr{C}$. The subspace of all nonatomic, σ-additive, totally finite, signed measures will be denoted by NA. The subspace of all σ-additive, totally finite, signed measures on $(I, \mathscr{C})$ will be denoted by M.

Let P be a subspace of BV. The set of all monotonic set functions in P is denoted P^+. A mapping of P into BV is called *positive* if it maps P^+ into BV^+. If P has no monotonic elements except 0, then every linear mapping is positive.

Let $\mathscr{I}$ denote the group of automorphisms of $(I, \mathscr{C})$, that is, the one-one functions from I onto I which are measurable in both directions. Each θ in $\mathscr{I}$ induces a linear mapping θ_* of BV onto itself, given by

$$(2.1) \qquad (\theta_* v)(S) = v(\theta S).$$

A subspace P of BV is called *symmetric* if $\theta_* P = P$ for each θ in $\mathscr{I}$.

The norm we shall use in BV is the *variation norm*, defined by

$$\| v \| = \inf \{ u(I) + w(I) \mid u - w = v, \text{ where } u \text{ and } w \text{ are monotonic} \}.$$

Unless otherwise stated the norm in BV will always be the variation norm; it is easily seen that it is indeed a norm.

A *chain* is a non-decreasing sequence of sets of the form

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = I.$$

A *link* of this chain is a pair of successive elements. A *subchain* is a set of links. A chain will be identified with the subchain consisting of all links. If v is a set function and Λ is a subchain of a chain Ω, then the *variation of v over* Λ is defined by

$$\| v \|_\Lambda = \Sigma \left| v(S_i) - v(S_{i-1}) \right|,$$

where the sum ranges over $\{i \mid \{S_{i-1}, S_i\} \in \Lambda\}$. For a fixed Λ, $\| \quad \|_\Lambda$ is a pseudonorm on BV, that is, it enjoys all the properties of a norm except that $\| v \| = 0$ does not necessarily imply that $v = 0$. In [1, Prop. 4.1] it is proved that $v \in BV$ iff $\| v \|_\Omega$ is bounded over all chains Ω, and if $v \in BV$ then

$$\| v \| = \sup_\Omega \| v \|_\Omega.$$

Clearly convergence in the variation norm implies pointwise convergence.

Let $v \in BV$ and $S \in \mathscr{C}$. Let v^S denote the restriction of v to the measurable subsets of S. Denote

$$|v|(S) = \| v^S \|.$$

This coincides with the usual concept of the total variation of a measure. It is easily seen that

$$|v|(S) = \sup \sum_{i=1}^{k} \left| v(S_i) - v(S_{i-1}) \right|$$

where the supremum is taken over all non-decreasing sequences

$$\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k = S.$$

It is clear that the variation norm coincides with the usual norm for bounded, finitely additive measures (see [2, 15, p. 140]). In [1, Prop. 4.3], it is shown that BV is complete in the variation norm. It can be proved in straightforward fashion that NA, M, and FA are closed in BV under the variation norm.

3. Statement of chief results

Absolute continuity of a set function with respect to another set function is defined in [1, Sect. 5] as follows. Let v and w be set functions; then v is absolutely continuous with respect to w (written $v \ll w$) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every chain Ω and every subchain Λ of Ω

$$\| w \|_{\Lambda} \leqq \delta \Rightarrow \| v \|_{\Lambda} \leqq \varepsilon.$$

Note that the relation is transitive, and that if v and w are measures, it coincides with the usual notion of absolute continuity.

A set function is said to be *absolutely continuous* if there is a measure $\mu \in NA^+$ such that $v \ll \mu$. The set of all absolutely continuous set functions in BV is denoted AC. AC is a closed linear subspace of BV ([1, Prop. 5.2]).

A weaker concept of continuity was introduced in [1, Proof of (44.27)]. Now we are going to define an even weaker concept of continuity. For simplicity, we will only define this continuity with respect to members of FA.

If $v \in BV$ and $\mu \in FA$ then v is said to be *weakly continuous with respect to* μ (written $v \underset{w}{<} \mu$) if for any S, $T \in \mathscr{C}$,

$$|\mu| (S \nabla T) = 0 \Rightarrow v(S) = v(T). \tag{3.1}$$

Note that $v \underset{w}{<} \mu$ and $\mu \underset{w}{<} \eta$ where $v \in BV$ and $\mu, \eta \in FA^+$ implies $v \underset{w}{<} \eta$.

A set function is said to be *weakly continuous* if there is a measure $\mu \in NA^+$ such that $v \underset{w}{<} \mu$. The set of all weakly continuous set functions in BV is denoted WC.

Measurable orders were introduced in [1, Sect. 12], for the purpose of trying to establish a notion of a value (in the game theoretic sense) based on random orders.

Intuitively, each order has a direction. (An order on I is a relation on I that is transitive, irreflexive and complete.) To emphasize this, orders are denoted $x \underset{\mathscr{R}}{<} y$ instead of the usual $x\mathscr{R}y$, the intuitive meaning being that x comes before y. The notation, $x \underset{\mathscr{R}}{\leqq} y$, $x \underset{\mathscr{R}}{>} y$ and $x \underset{\mathscr{R}}{\geqq} y$ are used similarly.

An *initial segment* is a set of the form $I(s, \mathscr{R}) = \{x \mid x \underset{\mathscr{R}}{<} s\}$ where $s \in I$. A *final segment* is a set of the form $E(s, \mathscr{R}) = \{x \mid x \underset{\mathscr{R}}{>} s\}$ where $s \in I$. An *initial set* is a set J which fulfills the condition $s \in J$, $s' \underset{\mathscr{R}}{\leqq} s \Rightarrow s' \in J$. An *$\mathscr{R}$-interval* is a set of the form $[s, t)_{\mathscr{R}} = \{x \mid s \underset{\mathscr{R}}{\leqq} x \underset{\mathscr{R}}{\leqq} t\}$ where s, $t \in I$. The entire space and the empty set are also considered as initial sets, and as such are denoted $I(\infty, \mathscr{R})$ and $I(-\infty, \mathscr{R})$ respectively. It is understood that $-\infty \underset{\mathscr{R}}{<} s \underset{\mathscr{R}}{<} \infty$ for each $s \in I$ and we denote $\{-\infty\} \cup I \cup \{\infty\}$ by $\bar{I}$. (Formally we extend $\mathscr{R}$ to $\bar{I}$. This however is a notational device; we are not adding anything to the underlying space, and all set functions and measures continue to be defined only on subsets of I.)

Denote by $F(\mathscr{R})$ the σ-field generated by all the initial segments. A *measurable* order is an order such that $F(\mathscr{R}) = \mathscr{C}$.

A set function v is called *orderable* [1, Sect. 12] if for each measurable order $\mathscr{R}$ there is a σ-additive measure $\varphi^{\mathscr{R}}v$ such that for all initial segments $I(s,\mathscr{R})$, we have

$$(\varphi^{\mathscr{R}}v)\,(I(s,\mathscr{R})) = v(I(s,\mathscr{R})). \tag{3.2}$$

Since (3.2) determines $(\varphi^{\mathscr{R}}v)$ on all the initial segments, and since by the measurability of $\mathscr{R}$ the initial segments generate $\mathscr{C}$, it follows that there can be at most one measure $\varphi^{\mathscr{R}}v$ satisfying (3.2). Thus for orderable set functions there is exactly one measure $\varphi^{\mathscr{R}}v$ satisfying (3.2). The set of all orerable set functions will be denoted *ORD*.

We are now ready to state the main results of this paper. (Proofs of Theorems 3.1, 3.2 and 3.3, and 3.4 and 3.4′ are given respectively in Sections 5, 6, and 7.)

THEOREM 3.1. *Let $v \in ORD$. Then v is nonatomic if and only if for any measurable order $\mathscr{R}$, $\varphi^{\mathscr{R}}v$ is nonatomic.*

THEOREM 3.2. *Let $v \in ORD$, $\mu \in M^+$. Then $v \underset{w}{<} \mu$ if and only if for any measurable order $\mathscr{R}$, $\varphi^{\mathscr{R}}v \underset{w}{<} \mu$.*

THEOREM 3.3. *Let $v \in ORD$, $\mu \in M^+$ and $v \underset{w}{<} \mu$. Then A is v-null if and only if for any measurable order $\mathscr{R}$, A is $\varphi^{\mathscr{R}}v$*-null.

For $v \in ORD$ write $K_v = \{\varphi^{\mathscr{R}}v \mid \mathscr{R}$ is a measurable order$\}$ and

$K'_v = \{|\varphi^{\mathscr{R}}v| \mid \varphi^{\mathscr{R}}v \in K_v\}$.

THEOREM 3.4. *Let $v \in ORD$ be nonatomic. Then $v \in AC$ if and only if K_v, (or equivalently, K'_v) is weak sequentially compact.*

THEOREM 3.4′. *Let $v \in ORD$. Then there exists a measure λ such that $v \ll \lambda$ i and only if K_v (or equivalently, K'_v) is weak sequentially compact.*

4. Some auxiliary properties of weak continuity and orderability of set functions

In this section we are going to investigate some properties of weak continuity and orderability of set functions.

LEMMA 4.1. *If $v \in BV$, $\mu \in FA$ then the following statements are equivalent:*

(i) $v \underset{w}{<} \mu$.

(ii) *If $S, T \in \mathscr{C}$ and $S \subset T$, then $|\mu|(S) = |\mu|(T) \Rightarrow v(S) = v(T)$.*

(iii) *If $S \in \mathscr{C}$ is μ-null then S is v-null.*

PROOF.

(i) $\Rightarrow$ (ii): this is immediate.

(ii) $\Rightarrow$ (iii): if S is μ-null then for any T, $|\mu|(T) = |\mu|(T \setminus (T \cap S))$, and (ii) now yields $v(T) = v(T \setminus S)$.

(iii) $\Rightarrow$ (i): If $|\mu|(S \nabla T) = 0$ then $S \nabla T$ is μ-null and therefore by (iii) v-null. Now

$$v(S) = v(S \setminus (S \nabla T)) = v(S \cap T) = v(T \setminus (S \nabla T)) = v(T). \qquad \text{Q.E.D.}$$

REMARK. Let μ be an n-dimensional, σ-additive measure whose components μ_i are in M^+. Let f be a real-valued function on the range of μ in E^n, with $f(0) = 0$, and $f \circ \mu \in BV$. Then $v \underset{w}{<} \Sigma |\mu_i|$.

PROPOSITION 4.2. *WC is a closed linear symmetric subspace of BV.*

PROOF. WC is easily seen to be linear. By definition $WC \subset BV$. The symmetry follows immediately from

$$v \underset{w}{<} \mu \Rightarrow \psi_* v \underset{w}{<} \psi_* \mu \quad \text{for each automorphism } \psi.$$

To prove WC is closed, let $\| v_i - v \| \to_{i \to \infty} 0$ where $v_i \underset{w}{<} \mu_i$ and $\mu_i \in NA^+$. Without loss of generality assume $\mu_i(I) = 1$ and set

$$\mu = \sum_{i=1}^{\infty} (\tfrac{1}{2})^i \mu_i.$$

Note that $\mu \in NA^+$, and $\mu_i \underset{w}{<} \mu$ for all i, hence $v_i \underset{w}{<} \mu$ for all i.

Let $S \subset T$ such that $\mu(S) = \mu(T)$. Now for a given $\varepsilon > 0$ let v_{i_0} be such that $\| v_{i_0} - v \| \leqq \frac{1}{2}\varepsilon$. Since $v_{i_0} \underset{w}{<} \mu$ we obtain

$$v_{i_0}(S) = v_{i_0}(T).$$

Now

$$\begin{aligned} |v(T) - v(S)| &\leqq |v(T) - v_{i_0}(T)| + |v_{i_0}(T) - v_{i_0}(S)| + |(v_{i_0}(S) - v(S)| \\ &\leqq 2\| v - v_{i_0} \| + 0 \leqq \varepsilon. \end{aligned}$$

ε was chosen arbitrarily, therefore $v(T) = v(S)$. Hence $v \underset{w}{<} \mu$ and $v \in WC$ is proved. Q.E.D.

If ψ is an automorphism of $(I, \mathscr{C})$, denote by $\psi\mathscr{R}$ the order defined by $\psi x \underset{\psi\mathscr{R}}{<} \psi y$ iff $x \underset{\mathscr{R}}{<} y$. Obviously, $\psi\mathscr{R}$ is measurable iff $\mathscr{R}$ is.

PROPOSITION 4.3.

(i) *ORD is a closed linear symmetric subspace of BV.*

(ii) *For all measurable orders* $\mathscr{R}$, $\varphi^{\mathscr{R}}$ *is a bounded linear operator on ORD. Moreover* $\| \varphi^{\mathscr{R}} \| = 1$.

(iii) $\varphi^{\mathscr{R}}(\psi_* v) = \psi_*(\varphi^{\psi\mathscr{R}} v)$.

PROOF. Obviously ORD is a linear subspace and for all measurable orders $\mathscr{R}$, $\varphi^{\mathscr{R}}$ is linear on ORD. Let $\mathscr{R}$ be a measurable order and let ψ be an automorphism of $(I, \mathscr{C})$. Then $\psi_* \varphi^{\psi\mathscr{R}} v$ is a σ-additive measure. Furthermore,

$$\psi(I(s, \mathscr{R})) = I(\psi s, \psi\mathscr{R})$$

and hence

$$\begin{aligned}(\psi_*(\varphi^{\psi\mathscr{R}} v))I(s, \mathscr{R})) &= (\varphi^{\psi\mathscr{R}} v)(\psi(I(s, \mathscr{R}))) \\ &= (\varphi^{\psi\mathscr{R}} v)(I(\psi s, \psi\mathscr{R})) = v(I(\psi s, \psi\mathscr{R})) \\ &= v(\psi(I(s, \mathscr{R}))) = (\psi_* v)(I(s, \mathscr{R})).\end{aligned}$$

Since $(\psi_* \varphi^{\psi\mathscr{R}} v)$ is a σ-additive measure satisfying (3.2) for $\psi_* v$ and $\mathscr{R}$, it follows $\varphi^{\mathscr{R}}(\psi_* v)$ exists and equals $\psi_*(\varphi^{\psi\mathscr{R}} v)$. Hence $\psi_* v \in ORD$.

Next, let $\mathscr{R}$ be a measurable order. In [1, Prop. 12.8] it has been proved that $\| \varphi^{\mathscr{R}} v \| \leqq \| v \|$, which shows that $\| \varphi^{\mathscr{R}} \| \leqq 1$. To see $\| \varphi^{\mathscr{R}} \| \geqq 1$ let μ be any probability measure on $\mathscr{C} = F(\mathscr{R})$, then clearly $\varphi^{\mathscr{R}} \mu = \mu \neq 0$. This shows $\| \varphi^{\mathscr{R}} \| = 1$.

Finally we will prove that ORD is closed. Let $\mathscr{R}$ be a measurable order and let $v_n \to v$ as $n \to \infty$, where $v_n \in ORD$ for each n. Then

$$\| \varphi^{\mathscr{R}} v_n - \varphi^{\mathscr{R}} v_m \| = \| \varphi^{\mathscr{R}}(v_n - v_m) \| \leqq \| (v_n - v_m) \|_{n,m \to \infty} \to 0.$$

Thus we see that $\varphi^{\mathscr{R}} v_n$ is a Cauchy sequence. As BV is complete in the variation norm [1, Sect. 4], it follows that $\varphi^{\mathscr{R}} v_n$ converges; we denote its limit by η. The set of σ-additive measures is closed in BV, and therefore η is a σ-additive measure. Moreover, convergence in the variation norm clearly implies pointwise convergence, therefore for every $s \in I$,

$$\eta(I(s, \mathscr{R})) = \lim_{n \to \infty} (\varphi^{\mathscr{R}} v_n)(I(s, \mathscr{R})) = \lim_{n \to \infty} v_n(I(s, \mathscr{R})) = v(I(s, \mathscr{R})).$$

Hence η is a σ-additive measure which fulfills (3.2). Hence $v \in ORD$ has been proved.

Q.E.D.

COROLLARY 4.4. *Let* $\mathscr{R}$ *be a measurable order. Then* $\varphi^{\mathscr{R}}$ *is positive on ORD.*

PROOF. If P is a linear subspace of BV and φ a linear operator from P into

$M \subset BV$ obeying the normalization condition $(\varphi^{\mathscr{R}} v)(I) = v(I)$ and $\|\varphi\| \leqq 1$, then φ is positive [1, Prop. 4.6]. This fact along with Proposition 4.3 complete the proof. Q.E.D.

PROPOSITION 4.5. *Let $v \in BV$ such that there is a measure μ for which $v \ll \mu$. Then $v \in ORD$.*

PROOF. See [1, Prop. 12.8], and note that the nonatomicity of μ was actually not used. Q.E.D.

5. Theorem 3.1

We are going to need several auxiliary lemmas for the proof of Theorem 3.1. Let us first start with a definition.

DEFINITION. A subset Q of I will be called *$\mathscr{R}$-dense* if for all $s, t \in I$ such that $s \underset{\mathscr{R}}{<} t$ there is a member $q \in Q$ such that $s \underset{\mathscr{R}}{\leqq} q \underset{\mathscr{R}}{\leqq} t$. By [1, Lem. 12.5], there exists a denumerable $\mathscr{R}$-dense set for any measurable order $\mathscr{R}$.

LEMMA 5.1. *Let $\mathscr{R}$ be a measurable order. Let $A \in \mathscr{C}$. Define an order $\mathscr{R}^*$ by*

$$x \underset{\mathscr{R}^*}{<} y \Leftrightarrow \begin{cases} x \in A,\ y \in A \text{ and } x \underset{\mathscr{R}}{<} y, \text{ or} \\ x \notin A,\ y \notin A \text{ and } x \underset{\mathscr{R}}{<} y, \text{ or} \\ x \notin A,\ y \in A. \end{cases}$$

(This means A is thrown beyond I/A and the order $\mathscr{R}$ is preserved on A and I/A.) Then $\mathscr{R}^*$ is measurable.

PROOF. The direction $F(\mathscr{R}^*) \subset \mathscr{C}$ is trivial. To prove the opposite we shall first show that there is a denumerable $\mathscr{R}^*$-dense set. Let Q be an $\mathscr{R}$-dense denumerable set. ([1, Lem. 12.5] assures its existence.) Denote

$$B_q = \begin{cases} I \setminus A & q \in A \\ A & q \in I \setminus A. \end{cases}$$

We have to substitute for q in B_q. If there is a minimal element in $E(q, \mathscr{R}) \cap B_q$ or a maximal element in $I(q, \mathscr{R}) \cap B_q$ then we might substitute it for q in B_q. Otherwise we might find a sequence which approaches in B_q the place where q has been. Denote by I_q the element or the sequence which substitute for q in B_q, then $Q' = (\cup_{q \in Q} I_q) \cup Q$ is a denumerable $\mathscr{R}^*$-dense set.

Let J be an $\mathscr{R}^*$-initial set; we shall show that $J \in F(\mathscr{R}^*)$. Denote

$$\bar{Q}' = Q' \cup \{-\infty\} \cup \{\infty\}, \qquad \bar{J} = J \cup \{-\infty\} \text{ and } J_1 = \bigcap_{q \in \bar{Q}' \setminus J} I(q, \mathscr{R}^*).$$

$J_1 \supset J$, and since Q' is $\mathscr{R}^*$ dense, it follows that $J_1 \setminus J$ contains at most two points. Since the intersection defining J_1 is denumerable it follows that $J_1 \in F(\mathscr{R}^*)$ and this assures $J \in F(\mathscr{R}^*)$.

Now clearly $I(x, \mathscr{R}) \in F(\mathscr{R}^*)$ for each x in I. This is sufficient to show $\mathscr{C} \subset F(\mathscr{R}^*)$ (recall $\mathscr{C} = F(\mathscr{R})$). Q.E.D.

Corollary 5.2. *Let $A_1, \cdots, A_n$ be distinct, measurable sets whose union is I. Then there is a measurable order $\mathscr{R}$ such that*

$$x \in A_i,\ y \in A_j, \text{ and } i < j \Rightarrow x \underset{\mathscr{R}}{<} y. \tag{5.1}$$

Proof. Start with the usual order on $[0,1]$ which is clearly measurable. Define an order which throws A_1 beyond $[0,1] \setminus A_1$ and preserves $\mathscr{R}$ on A_1 and $I \setminus A_1$. By repeating this for $A_2, \cdots, A_n$ we get an order which satisfies (5.1). Q.E.D.

Corollary 5.3. *Let $A_1, \cdots, A_n$ be distinct measurable sets whose union is I. Then there exists a measurable order $\mathscr{R}$ such that* (5.1) *is satisfied and there is an $\mathscr{R}$-minimal element in each A_i.*

Proof. Choose $x_i \in A_i$ for each i. Denote $B_{2i-1} = \{x_i\}$, $B_{2i} = A_i / \{x_i\}$, and apply Corollary 5.2, for the B_j's. Q.E.D.

We shall use the notation $A \underset{\mathscr{R}}{<} B$ when $x \underset{\mathscr{R}}{<} y$ for each $x \in A$ and $y \in B$.

Lemma 5.4. *Let $v \in ORD$ and let $S_1 \subset T_1 \subset S_2 \subset T_2 \subset \cdots \subset S_n \subset T_n$, where $S_i, T_i \in \mathscr{C}$ for $1 \leqq i \leqq n$. Then there is a measurable order $\mathscr{R}$ such that*

$$(\varphi^{\mathscr{R}} v)\left\{\bigcup_{i=1}^{n} (T_i \setminus S_i)\right\} = \sum_{i=1}^{n} v(T_i) - v(S_i).$$

Proof. Let us define

$$A_{2k} = T_k \setminus S_k, \quad A_{2k+1} = S_{k+1} \setminus T_k \quad \text{for } 1 \leqq k \leqq n,$$

and

$$A_{2n+1} = I \setminus T_n,\ A_0 = \varnothing,\ A_1 = S_1.$$

Corollary 5.3 assures the existence of a measurable order $\mathscr{R}$ such that

$$A_1 \underset{\mathscr{R}}{<} A_2 \underset{\mathscr{R}}{<} A_3 \underset{\mathscr{R}}{<} \cdots \underset{\mathscr{R}}{<} A_{2n+1},$$

and every set has an $\mathscr{R}$-first element which will be called x_i. Now

$$(\varphi^{\mathscr{R}} v)\left\{\bigcup_{i=1}^{n} (T_i \setminus S_i)\right\} = \sum_{i=1}^{n} (\varphi^{\mathscr{R}} v)([x_{2i}, x_{2i+1})_{\mathscr{R}})$$

$$= \sum_{i=1}^{n} v(I(x_{2i+1}, \mathscr{R})) - v(I(x_{2i}, \mathscr{R})) = \sum_{i=1}^{n} (v(T_i) - v(S_i)). \qquad \text{Q.E.D.}$$

We restate Theorem 3.1. Let $v \in ORD$. Then v is nonatomic if and only if for any measurable order $\mathscr{R}$, $\varphi^{\mathscr{R}} v$ is nonatomic.

REMARK. The conclusion need not hold if we do not assume $v \in ORD$, even if we do assume that for the $\mathscr{R}$ in question, there is a σ-additive measure $\varphi^{\mathscr{R}} v$ satisfying (3.2)! For example let $v = f \circ \lambda$, where λ is Lebesgue measure,

$$f(x) = \begin{cases} 0 & x \leqq \frac{1}{2} \\ 1 & x > \frac{1}{2}, \end{cases}$$

and let $\mathscr{R}$ be the usual order (which is obviously measurable); it is clear that $\varphi^{\mathscr{R}} v$ exists and equals the measure concentrated at $\frac{1}{2}$, which is not nonatomic. It can easily be shown that $v \notin ORD$, for denote by $\mathscr{R}'$ the order which throws $\frac{1}{2}$ beyond $[0,1]$ and coincides with the usual order on $[0,1] \setminus \{\frac{1}{2}\}$. $\mathscr{R}'$ is measurable (Lemma 5.1). If $\varphi^{\mathscr{R}'} v$ existed, then for $n \geqq 3$, $(\varphi^{\mathscr{R}'} v)([\frac{1}{2} - 1/n, \frac{1}{2} + 1/n)_{\mathscr{R}}) = 1$ in spite of the fact that $[\frac{1}{2} - 1/n, \frac{1}{2} + 1/n)_{\mathscr{R}'}$ is a decreasing sequence with a void intersection.

PROOF OF THEOREM 3.1. If v is not nonatomic, then there exists an $s \in I$ and a set $T \in \mathscr{C}$ such that $v(T \setminus \{s\}) \neq v(T)$. Looking at the chain $\emptyset \subset T \setminus \{s\} \subset T \subset I$ and using Lemma 5.4 we know that there exists a measurable order $\mathscr{R}$ such that $(\varphi^{\mathscr{R}} v)(T \setminus (T \setminus \{s\})) = v(T) - v(T \setminus \{s\}) \neq 0$, that is, $(\varphi^{\mathscr{R}} v)(\{s\}) \neq 0$. This contradicts the nonatomicity of $\varphi^{\mathscr{R}} v$.

For the opposite direction, let v in ORD be nonatomic, and $\mathscr{R}$ be a measurable order. $\varphi^{\mathscr{R}} v$ is σ-additive, therefore it is sufficient to prove $(\varphi^{\mathscr{R}} v)(\{s\}) = 0$ for all $s \in I$. Let $s \in I$ be fixed. Henceforth greater or smaller will be with respect to the order $\mathscr{R}$. Q will denote a denumerable $\mathscr{R}$-dense subset whose existence follows from [1, Prop. 12.4] and $\bar{Q} = \{-\infty\} \cup Q \cup \{\infty\}$.

Case (*i*). If there is an $\mathscr{R}$-minimal element in $E(s, \mathscr{R})$ (remember $E(s, \mathscr{R}) = \{x \mid x \underset{\mathscr{R}}{>} s\}$), let it be a. Then $\{s\} = [s, a)_{\mathscr{R}}$ and our conclusion becomes trivial. (This case holds when s is the greatest element in I, then $a = \infty$.)

Case (*ii*). In this case we assume that there exists no $\mathscr{R}$-maximal element in

$I(s,\mathscr{R})$, and no $\mathscr{R}$-minimal element in $E(s,\mathscr{R})$. W.l.o.g. we may assume that $s \notin Q$ (if $s \in Q$, change Q appropriately). Let

$$J = I(s,\mathscr{R}) \cup \{s\}, \qquad \bar{J} = J \cup \{-\infty\},$$
$$J_1 = \bigcap_{q \in \bar{Q}\setminus \bar{J}} I(q,\mathscr{R}) \text{ and } J_0 = \bigcup_{q \in \bar{Q}\cap \bar{J}} J(q,\mathscr{R}).$$

Then $J_1 \supset J \supset J_0$. The facts that Q is $\mathscr{R}$-dense and that there is no $\mathscr{R}$-maximal element in $I(s,\mathscr{R})$ imply that $J = J_0 \cup \{s\}$ and since there is no $\mathscr{R}$-minimal element in $E(s,\mathscr{R})$ and $s \notin Q$ it follows that $J_1 = J$, hence $J_1/J_0 = \{s\}$.

Since the intersection and union defining J_1 and J_0 respectively are denumerable, it follows that J_1 and J_0 are measurable. Furthermore since $I(q,\mathscr{R})$ are linearly ordered under inclusion each finite intersection equals one of the $I(q,\mathscr{R})$; hence

$$J_1 = \bigcap_{i=1}^{\infty} I(q_i^1,\mathscr{R})$$

where $\{q_i^1\}$ is an $\mathscr{R}$-decreasing sequence of points in $\bar{Q}\setminus \bar{J}$; that is, $I(q_i^1,\mathscr{R})$ is a decreasing set sequence. Similarly

$$J_0 = \bigcup_{i=1}^{\infty} I(q_i^0,\mathscr{R})$$

where $\{q_i^0\}$ is an $\mathscr{R}$-increasing sequence of points in $\bar{Q} \cap \bar{J}$; that is, $I(q_i^0,\mathscr{R})$ is an increasing set sequence. Obviously

$$J_1 \setminus J_0 = \bigcap_{i=1}^{\infty} I(q_i^1,\mathscr{R}) \setminus \bigcup_{i=1}^{\infty} I(q_i^0,\mathscr{R}) = \bigcap_{i=1}^{\infty} \{I(q_i^1,\mathscr{R}) \setminus I(q_i^0,\mathscr{R})\}.$$

Note that $I(q_i^1,\mathscr{R}) \setminus I(q_i^0,\mathscr{R})$ is a decreasing sequence and $\varphi^{\mathscr{R}}v$ is a totally finite σ-additive measure; this yields

$$\begin{aligned} (\varphi^{\mathscr{R}} v)\{s\} &= (\varphi^{\mathscr{R}}v)(J_1 \setminus J_0) = \lim_{i\to\infty} (\varphi^{\mathscr{R}}v)\{I(q_i^1,\mathscr{R}) \setminus I(q_i^0,\mathscr{R})\} \\ &= \lim_{i\to\infty} (v(I(q_i^1,\mathscr{R})) - v(I(q_i^0,\mathscr{R}))). \end{aligned} \tag{5.2}$$

Define an order $\mathscr{R}^*$ which throws s beyond all other elements and preserves $\mathscr{R}$ on $I\setminus\{s\}$. (Lemma 5.1 assures its measurability.) Denote

$$J_1^* = \bigcap_{i=1}^{\infty} I(q_i^1,\mathscr{R}^*), \qquad J_0^* = \bigcup_{i=1}^{\infty} I(q_i^0,\mathscr{R}^*).$$

Since $s \notin Q$, we obtain that for $i \geqq 1$

$$\{s\} \cup I(q_i^1,\mathscr{R}^*) = I(q_i^1,\mathscr{R}), \quad I(q_i^0,\mathscr{R}^*) = I(q_i^0,\mathscr{R});$$

this yields

$$\{s\} \cup J_1^* = J_1, \qquad J_0^* = J_0.$$

Now, since $J_1 = J = J_0 \cup \{s\}$ it follows that $J_1^* = J_0^*$, hence

$$\varnothing = J_1^* \setminus J_0^* = \bigcap_{i=1}^{\infty} I(q_i^1, \mathscr{R}^*) \setminus \bigcup_{i=1}^{\infty} I(q_i^0, \mathscr{R}^*)$$

$$= \bigcap_{i=1}^{\infty} I(q_i^1, \mathscr{R}^*) \setminus I(q_i^0, \mathscr{R}^*).$$

Since $I(q_i^1, \mathscr{R}^*) \setminus I(q_i^0, \mathscr{R}^*)$ is a decreasing sequence and $\varphi^{\mathscr{R}^*} v$ is a totally finite σ-additive measure we obtain

$$0 = (\varphi^{\mathscr{R}^*} v)(J_1^* \setminus J_0^*) = \lim_{i \to \infty} (\varphi^{\mathscr{R}^*} v)\{I(q_i^1, \mathscr{R}^*) \setminus I(q_i^0, \mathscr{R}^*)\}$$

$$= \lim_{i \to \infty} (v(I(q_i^1, \mathscr{R}^*)) - v(I(q_i^0, \mathscr{R}^*)))$$

$$= \lim_{i \to \infty} (v(I(q_i^1, \mathscr{R}) \setminus \{s\}) - v(I(q_i^0, \mathscr{R})))$$

$$= \lim_{i \to \infty} (v(I(q_i^1, \mathscr{R})) - v(I(q_i^0, \mathscr{R})))$$

$$= (\varphi^{\mathscr{R}} v)\{s\}.$$

We used (5.2) and the nonatomicity of v in the last two equalities.

Case (iii). In this case we assume there is an $\mathscr{R}$-maximal elementb_1 in $I(s, \mathscr{R})$, but there is no $\mathscr{R}$-minimal element in $E(s, \mathscr{R})$. W.l.o.g. assume again that $s \notin Q$. Let

$$J = I(s, \mathscr{R}) \cup \{s\} = I(b_1, \mathscr{R}) \cup \{b_1, s\}$$

$$\bar{J} = J \cup \{-\infty\}$$

$$J_1 = \bigcap_{q \in Q \setminus J} I(q, \mathscr{R}).$$

The same arguments used in the second case lead us to the conclusion that J_1 is measurable and $J_1 = \bigcap_{i=1}^{\infty} I(q_i^1, \mathscr{R})$, where q_i^1 is an $\mathscr{R}$- decreasing sequence. As there is no $\mathscr{R}$-minimal element in $E(s, \mathscr{R})$ and $s \notin Q$ it follows that $J_1 = J$. Note that $I(q_i^1, \mathscr{R})$ is a decreasing sequence of sets and $\varphi^{\mathscr{R}} v$ is a totally finite, σ-additive measure. Using the first case with respect to b_1 we obtain

$$(5.3) \qquad \begin{aligned} (\varphi^{\mathscr{R}} v)(\{s\}) &= (\varphi^{\mathscr{R}} v)(\{b_1, s\}) = (\varphi^{\mathscr{R}} v)(J_1) - (\varphi^{\mathscr{R}} v)(I(b_1, \mathscr{R})) \\ &= \lim_{i \to \infty} v(I(q_i^1, \mathscr{R})) - v(I(b_1, \mathscr{R})). \end{aligned}$$

Define a measurable order $\mathscr{R}_1$ which throws s beyond all other elements. (Lemma 5.1 assures its measurability.) Let

$$J_1^{(1)} = \bigcap_{i=1}^{\infty} I(q_i^1, \mathscr{R}_1).$$

Since $I(q_i^1, \mathscr{R}_1) \cup \{s\} = I(q_i^1, \mathscr{R})$ for all $i \geqq 1$ it follows that

$$J_1^{(1)} \cup \{s\} = J_1 = J.$$

Now, by using (5.3) and the nonatomicity of v we obtain that

$$\begin{aligned}(\varphi^{\mathscr{R}} v)\{s\} &= \lim_{i \to \infty} v(I(q_i^1, \mathscr{R})) - v(I(b_1, \mathscr{R})) \\ &= \lim_{i \to \infty} v(I(q_i^1, \mathscr{R}_1)) - v(I(b_1, \mathscr{R}_1)) \\ &= (\varphi^{\mathscr{R}_1} v)\left(\bigcap_{i=1}^{\infty} I(q_i^1, \mathscr{R}_1)\right) - (\varphi^{\mathscr{R}_1} v)\,(I(b_1, \mathscr{R})) \\ &= (\varphi^{\mathscr{R}_1} v)(J_1^{(1)} \backslash I(b_1, \mathscr{R}_1) = (\varphi^{\mathscr{R}_1} v)\{b_1\}.\end{aligned}$$

Clearly there is no $\mathscr{R}_1$-minimal element in $E(b_1, \mathscr{R}_1)$. If there is no $\mathscr{R}_1$-maximal element in $I(b_1, \mathscr{R}_1)$ (or equivalently in $I(b_1, \mathscr{R})$), then the proof of the second case yields that

$$0 = (\varphi^{\mathscr{R}_1} v)\{b_1\} = (\varphi^{\mathscr{R}} v)\{s\}.$$

If there is such an element, we denote it by b_2. We define an order $\mathscr{R}_2$ which throws b_1 beyond all other elements, and conclude analogously that

$$(\varphi^{\mathscr{R}_2} v)(\{b_2\}) = (\varphi^{\mathscr{R}_1} v)\,(\{b_1\}) = (\varphi^{\mathscr{R}} v)\,(\{s\}).$$

Going on in this way we build a sequence of measurable orders $\mathscr{R}_n$ and a sequence of elements in I denoted b_n, such that $\mathscr{R}_{n+1}$ is the measurable order which is obtained from $\mathscr{R}_n$ by throwing b_n beyond all other elements and b_{n+1} is the $\mathscr{R}_n$-maximal element in $I(b_n, \mathscr{R}_n) = I(b_n, \mathscr{R})$ (provided it exists). We obtain

$$(\varphi^{\mathscr{R}_{n+1}})(\{b_{n+1}\}) = (\varphi^{\mathscr{R}_r})(\{b_n\}) = (\varphi^{\mathscr{R}} v)(\{s\}).$$

If at some step there is no $\mathscr{R}_n$-maximal element in $I(b_n, \mathscr{R}_n) = I(b_n, \mathscr{R})$, the proof of the second case yields $0 = (\varphi^{\mathscr{R}_n} v)(\{b_n\}) = (\varphi^{\mathscr{R}} v)\,(\{s\})$. If for some n, $b_n = -\infty$ we easily obtain $0 = (\varphi^{\mathscr{R}_n} v)(\{b_n\}) = (\varphi^{\mathscr{R}} v)(\{s\})$. Otherwise the procedure goes on for all $n \geqq 1$. In this case define an order $\mathscr{R}^*$

$$x \underset{\mathscr{R}^*}{<} y \Leftrightarrow \begin{cases} x, y \in \{b_n\}_{n \geqq 1} \text{ and } y \underset{\mathscr{R}}{<} x, \\ \text{otherwise and } x \underset{\mathscr{R}}{<} y. \end{cases}$$

This means we reverse the order $\mathscr{R}$ on $\{b_n\}_{n \geqq 1}$ and preserve it everywhere else. Note that $\mathscr{R}^*$ is measurable and that $\{s\} = J_1 \setminus I(s, \mathscr{R}) = J_1 \setminus I(s, \mathscr{R}^*)$. Clearly

$$(\varphi^{\mathscr{R}^*} v)(J_1) = \lim_{i \to \infty} (\varphi^{\mathscr{R}^*} v)(I(q_i^1, \mathscr{R}^*)) = \lim_{i \to \infty} (\varphi^{\mathscr{R}} v)(I(q_i^1, \mathscr{R})),$$

hence

$$\begin{aligned} (\varphi^{\mathscr{R}^*} v)(\{s\}) &= (\varphi^{\mathscr{R}^*} v)(J_1) - (\varphi^{\mathscr{R}^*} v)(I(s, \mathscr{R}^*)) \\ &= \lim_{i \to \infty} (\varphi^{\mathscr{R}} v)(I(q_i^1, \mathscr{R})) - v(I(s, \mathscr{R}^*)) \\ &= (\varphi^{\mathscr{R}} v)(J_1) - v(I(s, \mathscr{R})) \\ &= (\varphi^{\mathscr{R}} v)(J_1) - (\varphi^{\mathscr{R}} v)(I(s, \mathscr{R})) \\ &= (\varphi^{\mathscr{R}} v)(\{s\}). \end{aligned}$$

Clearly there is no $\mathscr{R}^*$-minimal element in $E(s, \mathscr{R}^*)$, and no $\mathscr{R}^*$-maximal element in $I(s, \mathscr{R}^*)$; therefore the second case assures that

$$0 = (\varphi^{\mathscr{R}^*} v)(\{s\}) = (\varphi^{\mathscr{R}} v)(\{s\}).$$

Q.E.D.

6. Theorems 3.2. and 3.3.

We start this section with the proof of Proposition 6.1 which is a slight variation of Theorem 3.1.

PROPOSITION 6.1. *Let* $v \in ORD$, $\mu \in M^+$ *and* $v \underset{w}{<} \mu$. *Then* $\{s\}$ *is* v*-null* $\Leftrightarrow (\varphi^{\mathscr{R}} v)(\{s\}) = 0$ *for each measurable order* $\mathscr{R}$.

REMARK. This proposition gives a sufficient condition that an element of I is not an atom for any $\varphi^{\mathscr{R}} v$, (where $\mathscr{R}$ are measurable orders). From this point of view the proposition is analogous to Theorem 3.1.

PROOF. The proof is similar to what we have done in the proof of Theorem 3.1 (see Section 5); the trivial direction is the same, the other is similarly divided into three cases.

Cases (*i*) *and* (*ii*). The proof is just as in Theorem 3.1, except that the last sentence of Case (ii) should read: "We used (5.2) and the fact that $\{s\}$ is v-null".

Case (*iii*). Let q_i^1, J be as in the proof of the third case in Theorem 3.1. We easily obtain that

$$(6.1) \qquad (\varphi^{\mathscr{R}}v)(\{s\}) = (\varphi^{\mathscr{R}}v)(J_1) - (\varphi^{\mathscr{R}}v)(I(s,\mathscr{R})) = \lim_{i\to\infty} v(I(q_i^1,\mathscr{R})) - v(I(s,\mathscr{R})).$$

Now, choose any sequence $\{a_i|\ i \geqq 1\}$ consisting of elements of I such that $\mu(\{a_i\}) = 0$ for all $i \geqq 1$, and $\{a_i | i \geqq 1\} \cap \{q_i^1 | i \geqq 1\} = \emptyset$. Clearly there exists such a sequence. Define an order $\mathscr{R}^*$ that puts the sequence $\{a_i | i \geqq 1\}$ just after s and $a_i \underset{\mathscr{R}^*}{<} a_j$ iff $i < j$, that means

$$x \underset{\mathscr{R}^*}{<} y \Leftrightarrow \begin{cases} x, y \notin \{a_i\} & \text{and } x \underset{\mathscr{R}}{<} y, \\ x \in \{a_i\},\ y \notin \{a_i\} & \text{and } s \underset{\mathscr{R}}{<} y, \\ y \in \{a_i\}, x \notin \{a_i\} & \text{and } x \underset{\mathscr{R}}{<} s, \\ x = a_i,\ x = a_j & \text{and } i < j. \end{cases}$$

Note that $\mathscr{R}^*$ is a measurable order. Since $\mu(\{a_i\}) = 0$ for all i we obtain that $\{a_i|\ i \geqq 1\}$ is μ-null, hence it is v-null and therefore

$$t \notin \{a_i|\ i \geqq 1\} \Rightarrow v(I(t,\mathscr{R})) = v(I(t,\mathscr{R}^*)).$$

This holds in particular for $t = q_i^1$ $(i \geqq 1)$. Now note that for all $i \geqq 1$, $v(I(s,\mathscr{R}) = v(I(a_i,\mathscr{R}^*))$; hence we obtain, using (6.1), that

$$\begin{aligned} (\varphi^{\mathscr{R}}v)(\{s\}) &= \lim_{i\to\infty} v(I(q_i^1,\mathscr{R})) - v(I(s,\mathscr{R})) \\ &= \lim_{i\to\infty} v(I(q_i^1,\mathscr{R}^*)) - \lim_{i\to\infty} v(I(a_i,\mathscr{R}^*)) \\ &= (\varphi^{\mathscr{R}^*}v)(J_1 \cup \{a_i | i \geqq 1\}) - \lim_{i\to\infty} (\varphi^{\mathscr{R}^*}v)(I(a_i,\mathscr{R}^*)) \\ &= (\varphi^{\mathscr{R}^*}v)(J_1 \cup \{a_i | i \geqq 1\}) - (\varphi^{\mathscr{R}^*}v)(J_1 \cup \{a_i | i \geqq 1\} = 0. \end{aligned}$$

Q.E.D.

REMARK. Note that under the assumptions of Proposition 6.1, we could not repeat the proof of Case (iii) in Theorem 3.1 since we could not know whether b_1 is v-null or not. On the other hand, we could not use there the idea we use here, since the monotonicity of v does not necessarily imply the existence of a denumerable v-null set.

COROLLARY 6.2. *Let $v \in ORD$, $\mu \in M^+$ and $v \underset{\infty}{<} \mu$. Let A be a denumerable set in $\mathscr{C}$. Then $\mu(A) = 0 \Rightarrow |\varphi^{\mathscr{R}} v|(A) = 0$ for all measurable orders $\mathscr{R}$.*

PROOF. $\mu(A) = 0$ implies $\mu(\{s\}) = 0$ for all s in A. Hence all s in A are v-null. Now by Proposition 6.1 we obtain $(\varphi^{\mathscr{R}}v)(\{s\}) = 0$ for all s in A which easily completes our proof. Q.E.D.

We restate Theorem 3.2. Let $v \in ORD$, $\mu \in M^+$. Then $v \underset{w}{<} \mu \Leftrightarrow \varphi^{\mathscr{R}}v \underset{w}{<} \mu$ for each measurable order $\mathscr{R}$.

REMARK. If $v \notin ORD$ we cannot assure $v \underset{w}{<} \mu \Rightarrow \varphi^{\mathscr{R}}v \underset{w}{<} \mu$ even if $\varphi^{\mathscr{R}}v$ happens to be defined (by (3.2)). Indeed, let λ be Lebesgue measure and let $v = f \circ \lambda$, where

$$f(x) = \begin{cases} 0 & x \leqq \frac{1}{2} \\ 1 & x > \frac{1}{2}. \end{cases}$$

If $\mathscr{R}$ is the usual order on $[0,1]$, then $\varphi^{\mathscr{R}}v$ is the measure concentrated at $\frac{1}{2}$; now $v \underset{w}{<} \lambda$, but $\varphi^{\mathscr{R}}v \underset{w}{<} \lambda$ does not hold.

PROOF. For the implication $\Leftarrow$, if $v \underset{w}{<} \mu$ does not hold then there exist S, $T \in \mathscr{C}$, $S \subset T$ where $\mu(T) = \mu(S)$ and $v(T) \neq v(S)$. Looking at the chain $\emptyset \subset S \subset T \subset I$ and using Lemma 5.4, we know that there exists a measurable order $\mathscr{R}$ such that $(\varphi^{\mathscr{R}}v)(T \setminus S) = v(T) - v(S) \neq 0$. This contradicts the assumption that $\varphi^{\mathscr{R}}v \underset{w}{<} \mu$.

For the implication $\Rightarrow$, $\varphi^{\mathscr{R}}v \underset{w}{<} \mu$ means usual absolute continuity of a measure with respect to another measure (see [2, p. 131]).

Let $H(\mathscr{R})$ be the field (not σ-field) generated by the initial segments $I(s;\mathscr{R})$. If two finitely additive measures μ_1, λ_1 are defined on a field Σ_1 and there exist countably additive extensions μ_2, λ_2, of μ_1, λ_1 respectively to the σ-field generated by Σ_1, then λ_1 is μ_1-continuous iff λ_2 is μ_2-continuous [2, IV 9–13, p. 315]. $\varphi^{\mathscr{R}}v$ and μ are extensions to $\mathscr{C}$ of $\varphi^{\mathscr{R}}v \mid H(\mathscr{R})$ and $\mu \mid H(\mathscr{R})$ respectively. If $\varphi^{\mathscr{R}}v \mid H(\mathscr{R})$ is not $\mu \mid H(\mathscr{R})$-continuous then there is an ε such that for each $n \geqq 1$ there exists a finite sequence

$$s_1^{(n)} \underset{\mathscr{R}}{<} t_1^{(n)} \underset{\mathscr{R}}{<} s_2^{(n)} \cdots s_{k_n}^{(n)} \underset{\mathscr{R}}{<} t_{k_n}^{(n)}$$

such that

$$\mu\left\{\bigcup_{i=1}^{k_n} [s_i^{(n)}, t_i^{(n)})_{\mathscr{R}}\right\} < \frac{1}{2^n} \text{ and } \left|(\varphi^{\mathscr{R}}v)\left\{\bigcup_{i=1}^{k_n} [s_i^{(n)}, t_i^{(n)})_{\mathscr{R}}\right\}\right| > \varepsilon.$$

Denoting

$$W_i^{(n)} = [s_i^{(n)}, t_i^{(n)})_{\mathscr{R}}, \qquad W_n = \bigcup_{i=1}^{k_n} W_i^{(n)},$$

$$A_n = \bigcup_{k=n}^{\infty} W_k \quad \text{and} \quad A = \bigcap_{n=1}^{\infty} A_n,$$

we obtain that

$$\mu(A_n) \leqq \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}$$

$$|\varphi^{\mathscr{R}}v|(A_n) \geqq |\varphi^{\mathscr{R}}v|(W_n) \geqq |(\varphi^{\mathscr{R}}v)(W_n)| > \varepsilon.$$

Note that $\{A_n\}$ is a decreasing sequence and $\bigcap_{n=1}^{\infty} A_n = A$; hence by the total finiteness of μ and $\varphi^{\mathscr{R}}v$

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) = 0$$

$$|\varphi^{\mathscr{R}}v|(A) = \lim_{n\to\infty} |\varphi^{\mathscr{R}}v|(A_n) > \varepsilon.$$

If it happens that A is empty or denumerable, then we arrive at a contradiction to Corollary 6.2, which completes the proof in this case.

If A is non denumerable we still know that $\mu(A) = 0$, and since $v \underset{w}{<} \mu$, it follows that A is v-null.

Let us define an order $\mathscr{R}^*$ that throws

$$A' = A \setminus \{s_i^{(n)}, t_i^{(n)} \mid 1 \leqq i \leqq k_n, n \geqq 1\}$$

beyond $I \setminus A'$ and preserves $\mathscr{R}$ on A' and $I \setminus A'$. From Lemma 5.1 it follows that $\mathscr{R}^*$ is measurable. Define

$$W_i^{(n)*} = [s_i^{(n)}, t_i^{(n)}]_{\mathscr{R}^*}, \quad W_n^* = \bigcup_{i=1}^{k_n} W_i^{(n)*},$$

$$A_n^* = \bigcup_{k=n}^{\infty} W_k^* \quad \text{and} \quad A^* = \bigcap_{n=1}^{\infty} A_n^*.$$

Clearly $I(a,\mathscr{R}^*) = I(a,\mathscr{R}) \setminus A'$ for each $a \in I \setminus A'$, in particular for $a = s_i^{(n)}$, $t_i^{(n)}$. Therefore

$$W_i^{(n)*} = W_i^{(n)} \setminus A', \qquad W_n^* = W_n \setminus A',$$

$$A_n^* = A_n \setminus A' \quad \text{and} \quad A^* = A \setminus A'.$$

Since A is v-null and $A' \subset A$, A' is v-null; therefore

$$\begin{aligned}(\varphi^{\mathscr{R}^*}v)(W_i^{(n)*}) &= (\varphi^{\mathscr{R}^*}v)([s_i^{(n)}, t_i^{(n)})_{\mathscr{R}^*}) \\ &= v(I(t_i^{(n)}, \mathscr{R}^*) - v(I(s_i^{(n)}, \mathscr{R}^*)) \\ &= v(I(t_i^{(n)}, \mathscr{R})) - v(I(s_i^{(n)}, \mathscr{R})) \\ &= (\varphi^{\mathscr{R}}v)([s_i^{(n)}, t_i^{(n)})_{\mathscr{R}}) \\ &= (\varphi^{\mathscr{R}}v)(W_i^{(n)}).\end{aligned}$$

Hence,

$$\left|(\varphi^{\mathscr{R}^*}v)\,(W_n^*)\right| = \left|\sum_{i=1}^{k_n}(\varphi^{\mathscr{R}^*}v)\,(W_i^{(n)*})\right|$$

$$= \left|\sum_{i=1}^{k_n}(\varphi^{\mathscr{R}}v)(W_i^{(n)})\right| = \left|(\varphi^{\mathscr{R}}v)(W_n)\right| > \varepsilon.$$

Note that $\{A_n^*\}$ is a decreasing sequence and $\bigcap_{n=1}^{\infty}A_n = A^*$; hence by the total positivity of $\varphi^{\mathscr{R}^*}v$

$$|\varphi^{\mathscr{R}^*}v|(A^*) = \liminf_{n\to\infty}|\varphi^{\mathscr{R}^*}v|(A_n^*) \geqq \liminf_{n\to\infty}|\varphi^{\mathscr{R}^*}v|\,(W_n^*) \geqq \liminf_{n\to\infty}\left|(\varphi^{\mathscr{R}^*}v)(W_n^*)\right| > \varepsilon.$$

By the non-negativity of μ and the fact that $A^* \subset A$, it follows that

$$0 \leqq \mu(A^*) \leqq \mu(A) = 0.$$

Since A^* is contained in $\{s_i^{(n)}, t_i^{(n)} \mid 1 \leqq i \leqq k_n,\ n \geqq 1\}$, it is denumerable; hence we get a contradiction to Corollary 6.2. This completes the proof of Theorem 3.2.

Q.E.D.

We restate Theorem 3.3. Let $v \in ORD$, $\mu \in M^+$ and $v \underset{w}{<} \mu$. Then A is v-null $\Leftrightarrow A$ is $\varphi^{\mathscr{R}}v$-null for all measurable orders $\mathscr{R}$.

Remark. If $v \notin ORD$, the conclusion need not follow even if $\varphi^{\mathscr{R}}v$ happens to be defined (by (3.2)). Take the example in the remark following the restatement of Theorem 3.2; then $(\varphi^{\mathscr{R}}v)(\{\tfrac{1}{2}\}) = 1$ but $\tfrac{1}{2}$ is a v-null set.

Proof of Theorem 3.3. (Aumann). For the implication $\Leftarrow$, if A is $(\varphi^{\mathscr{R}}v\text{-})$null for each measurable order $\mathscr{R}$ then $\left|\varphi^{\mathscr{R}}v\right|(A) = 0$ for each $\mathscr{R}$. Assume there exists a $B \in \mathscr{C}$ such that $v(B) \neq v(B\setminus A)$. Looking at the chain $\varnothing \subset B\setminus A \subset B \subset I$ Lemma 5.4 yields the existence of a measurable order $\mathscr{R}$ such that

$$(\varphi^{\mathscr{R}}v)(B \cap A) = (\varphi^{\mathscr{R}}v)(B\setminus(B\setminus A)) = v(B) - v(B\setminus A) \neq 0;$$

hence $(\varphi^{\mathscr{R}}v)\,(B \cap A) \neq 0$, which contradicts the fact that $\left|\varphi^{\mathscr{R}}v\right|(A) = 0$.

For the implication $\Rightarrow$, let $v \underset{w}{<} \mu$ where $\mu \in M^+$. Let A be v-null and $\mathscr{R}$ be a measurable order. We shall show that $\left|\varphi^{\mathscr{R}}v\right|(A) = 0$.

Define a measure $\mu_A(S) = \mu(S\setminus A)$; then $\mu_A \in M^+$. We shall show that $v \underset{w}{<} \mu_A$. Let $U \subset T$ be such that $\mu_A(T\setminus U) = 0$, which means $\mu((T\setminus U)\setminus A) = 0$. Since $v \underset{w}{<} \mu$, it follows that $(T\setminus U)\setminus A$ is v-null. Remembering that A is v-null, we obtain

$$\left|v(T)-v(U)\right| = \left|v(T)-v(T\setminus A)\right| + \left|v(T\setminus A)-v(U\setminus A)\right| + \left|v(U\setminus A)-v(U)\right|$$

$$= \left|v(T\setminus A)-v((T\setminus A)\setminus((T\setminus U)\setminus A))\right| = 0.$$

Now, $v \underset{w}{\lesssim} \mu_A$, $\mu_A \in M^+$ and Theorem 3.2 imply that $\varphi^{\mathscr{R}}v \underset{w}{\lesssim} \mu_A$ for each measurable order $\mathscr{R}$. We know that

$$\mu_A(A) = \mu(A\setminus A) = 0;$$

therefore $(\varphi^{\mathscr{R}}v)(A) = 0$. Q.E.D

7. Theorem 3.4

Recall that for $v \in ORD$, $K_v = \{\varphi^{\mathscr{R}}v \mid \mathscr{R}$ is a measurable order$\}$ and $K'_v = \{|\varphi^{\mathscr{R}}v| \mid \varphi^{\mathscr{R}}v \in K_v\}$.

We restate Theorem 3.4. Let $v \in ORD$ be nonatomic. Then $v \in AC$ iff K_v (or equivalently K'_v) are weak sequentially compact (henceforth abbreviated wsc).

REMARK. A set S is wsc in a Banach space X iff it is wsc in any fixed, closed linear subspace of X that contains S. One direction is immediate, the other follows from the Hahn-Banach theorem. Hence we do not have to mention in which space K_v and K'_v are wsc.

LEMMA 7.1. *A set K of σ-additive measures is* wsc *iff* $K' = \{|\mu| \mid \mu \in K\}$ *is* wsc.

PROOF OF THEOREM 3.4. A *measure* in this section means *σ-additive, totally finite, signed measure.* A necessary and sufficient condition for weak sequential compactness of a set K of measures [2, Th. IV.9.2, p. 306] is that K is bounded and that there exists a positive measure λ such that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\lambda(E) \leqq \delta$ for $E \in \mathscr{C}$ implies $\mu(E) \leqq \varepsilon$ for all $\mu \in K$. (The last condition will be denoted $\mu(E) \to_{\lambda(E)\to 0} 0$ uniformly with respect to μ in K.) Clearly the boundedness condition is equivalent in K and K' (since $\|\mu\| = \||\mu|\|$). Noting that $|\mu(E)| \leqq |\mu|(E)$ for all $E \in \mathscr{C}$ completes the proof that if K' is wsc then K is wsc.

Finally let K be wsc and λ be the positive measure in the above condition. For $\varepsilon > 0$, let δ be such that

$$\lambda(E) \leqq \delta \Rightarrow |\mu(E)| \leqq \tfrac{1}{2}\varepsilon \text{ for all } \mu \in K.$$

Since λ is positive, we obtain that for all $F \subset E$,

$$\lambda(E) \leqq \delta \Rightarrow |\mu(F)| + |\mu(E\setminus F)| \leqq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Hence for all $\mu \in K$,

$$\lambda(E) \leqq \delta \Rightarrow |\mu|(E) = \sup_{F \subset E} \{|\mu(F)| + |\mu(E\setminus F)|\} \leqq \varepsilon. \qquad \text{Q.E.D.}$$

PROOF OF THEOREM 3.4. Let K_v be wsc. Then by [2, IV.9.2, p. 306] we know the existence of a positive measure λ such that $(\varphi^{\mathscr{R}}v)(E) \to 0$ as $\lambda(E) \to 0$ uniformly with respect to $\varphi^{\mathscr{R}}v \in K_v$. If we follow the proof of that theorem we find that λ is defined as the sum of a series of modified members of the weak sequentially compact set. As v is nonatomic and $v \in ORD$, each $\varphi^{\mathscr{R}}_v$ is nonatomic (Theorem 3.1); hence all $\varphi^{\mathscr{R}}v$'s are nonatomic, and therefore we might demand that λ be nonatomic. So let us assume $\lambda \in NA^+$.

We are going to show $v \ll \lambda$. If this assumption does not hold, then there exists $\varepsilon_0 > 0$ such that for any given integer $n \geqq 1$ there is a chain

$$\varnothing \subset S_1^{(n)} \subset T_1^{(n)} \subset \cdots \subset S_{k_n}^{(n)} \subset T_{k_n}^{(n)} \subset I$$

such that

$$\sum_{i=1}^{k_n} |\lambda(T_i^{(n)}) - \lambda(S_i^{(n)})| \leqq \frac{1}{n}$$

and

$$\sum_{i=1}^{k_n} |v(T_i^{(n)}) - v(S_i^{(n)})| \geqq \varepsilon_0.$$

Clearly, by omitting some sets and changing indices, we may demand that

$$\left|\sum_{i=1}^{k_n} (v(T_i^{(n)}) - v(S_i^{(n)}))\right| \geqq \tfrac{1}{2}\varepsilon_0.$$

Set

$$A_n = \bigcup_{i=1}^{n} (T_i^{(n)} \setminus S_i^{(n)}).$$

Since λ is positive,

$$\lambda(A_n) = \sum_{i=1}^{k_n} |\lambda(T_i^{(n)}) - \lambda(S_i^{(n)})| \leqq \frac{1}{n}.$$

Lemma 5.4 yields the existence of measurable orders $\mathscr{R}_n$ such that

$$(\varphi^{\mathscr{R}_n}v)(A_n) = (\varphi^{\mathscr{R}_n}v)\left\{\bigcup_{i=1}^{k_n} (T_i^{(n)} \setminus S_i^{(n)})\right\} = \sum_{i=1}^{k_n} (v(T_i^{(n)}) - v(S_i^{(n)})).$$

Then

$$\left|(\varphi^{\mathscr{R}_n}v)(A_n)\right| = \left|\sum_{i=1}^{k_n}(v(T_i^{(n)}) - v(S_i^{(n)}))\right| \geqq \tfrac{1}{2}\varepsilon_0.$$

This, and the fact that $\lambda(A_n) \leqq 1/n$ contradicts the uniform limit $(\varphi^{\mathscr{R}}v)(E) \to 0$ as $\lambda(E) \to 0$ with respect to $\varphi^{\mathscr{R}}v \in K_v$; hence $v \ll \lambda$.

Now let $v \in AC$. To prove that K_v is wsc it is sufficient to show that K_v is bounded and that there exists a measure $\lambda \in M^+$ such that $(\varphi^{\mathscr{R}}v)(E) \to 0$ as $\lambda(E) \to 0$ uniformly with respect to $\varphi^{\mathscr{R}}v \in K_v$. K_v is clearly bounded since $\|\varphi^{\mathscr{R}}v\| \leqq \|v\|$ for all measurable orders (Proposition 4.3). To prove the uniform limit let λ be the measure in NA^+ such that $v \ll \lambda$, let $\varepsilon > 0$ be given, and let $\delta > 0$ correspond to $\frac{1}{2}\varepsilon$ in accordance with the definition $v \ll \lambda$. Let $\mathscr{R}$ be a measurable order and let $H(\mathscr{R})$ be the field (not σ-field) generated by the initial segments. Clearly a set in $H(\mathscr{R})$ is of the form $U = \bigcup_{i=1}^{n}[s_i, t_i)_{\mathscr{R}}$, where

$$s_1 \underset{\mathscr{R}}{<} t_1 \underset{\mathscr{R}}{<} s_2 \underset{\mathscr{R}}{<} \cdots \underset{\mathscr{R}}{<} s_n \underset{\mathscr{R}}{<} t_n.$$

Look at the subchain Λ consisting of the links $\{I(s_i, \mathscr{R}), I(t_i, \mathscr{R})\}$. Then

$$\|\lambda\|_\Lambda = \sum_{i=1}^{n}\left|\lambda(I(t_i, \mathscr{R})) - \lambda(I(s_i, \mathscr{R}))\right| = \lambda(U)$$

and

$$\|v\|_\Lambda = \sum_{i=1}^{n}\left|v(I(t_i, \mathscr{R})) - v(I(s_i, \mathscr{R}))\right|$$

$$\geqq \left|\sum_{i=1}^{n}(v(I(t_i, \mathscr{R})) - v(I(s_i, \mathscr{R})))\right| = \left|(\varphi^{\mathscr{R}}v)(U)\right|.$$

Hence

$$\lambda(U) \leqq \delta \Rightarrow \|\lambda\|_\Lambda \leqq \delta \Rightarrow \|v\|_\Lambda \leqq \tfrac{1}{2}\varepsilon \Rightarrow \left|(\varphi^{\mathscr{R}}v)(U)\right| \leqq \tfrac{1}{2}\varepsilon. \tag{7.1}$$

Now by a standard approximation theorem (one uses [3, Th. D, p. 56] on the measure $\mu + |\varphi^{\mathscr{R}}v|$), every measurable set can be approximated by members of $H(\mathscr{R})$ simultaneously with respect to μ and with respect to $|\varphi^{\mathscr{R}}v|$. Hence if $\mu(S) < \delta$, there exists a $U \in H(\mathscr{R})$ such that $\mu(U) \leqq \delta$ and $|\varphi^{\mathscr{R}}v|\ (U \triangledown S) \leqq \frac{1}{2}\varepsilon$. This and (7.1) imply that if $\mu(S) < \delta$ then $\left|(\varphi^{\mathscr{R}}v)\ (S)\right| \leqq \varepsilon$. Hence

$$\mu(S) \leqq \tfrac{1}{2}\delta \Rightarrow (\varphi^{\mathscr{R}}v)(S)| \leqq \varepsilon$$

for all measurable orders $\mathscr{R}$. This completes the proof that K_v is wsc. Q.E.D.

We restate Theorem 3.4′. Let $v \in ORD$. Then there exists a measure λ such that $v \ll \lambda$ iff K_v (or equivalently K'_v) is wsc.

PROOF OF THEOREM 3.4′. We proceed exactly as in the proof of Theorem 3.4, replacing $\lambda \in NA^+$ by $\lambda \in M^+$ and omitting the part which proves the nonatomicity of λ.

Q.E.D.

ACKNOWLEDGEMENTS

My deepest thanks are given to my teacher and adviser Professor R. J. Aumann for his guidance and invaluable assistance during the preparation of this work. I would also like to thank Miss Gail Lemmond for her skillful work in typing this paper.

REFERENCES

1. R. J. Aumann and L. S. Shapley, *Values of Non-atomic Games*, Princeton University Press, Princeton, New Jersey, 1973.
2. N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1958.
3. P. R. Halmos, *Measure Theory*, Van Nostrand, Princeton, 1950.
4. U. G. Rothblum, *Orderable Set Functions and Continuity III* (to appear).

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